Measuring Transformation Error by RMS Deviation

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Abstract

This report describes the RMS deviation measure used to calculate the average difference between two affine transformations. The measure is easy to calculate (a few simple matrix products and a trace) and gives an estimate in real units (mm) of the difference in transformations.

1 Introduction

It is often necessary to measure the difference between two transformations in order to assess the accuracy and robustness of registration methods. This measurement should not be biased by the selection of certain key points (such as the corners of a cube). One such measure is the RMS deviation measure which:

- can be calculated by a simple formula,
- is the average error over the expected brain volume (taken to be a sphere).

2 Transformations

A transformation from one volume to another can act either in voxel coordinates, world coordinates or a combination of both. In general, voxel and world coordinates are related by a simple affine transformation $T_W : V \rightarrow W$, where V is the voxel coordinate space, and W the world coordinate space. By using this transformation, all transformations can be expressed as transformations between world coordinates, which is more convenient and will be assumed from here on.

For example, if $T : V_2 \rightarrow V_1$ is a transformation between two voxel coordinate spaces, then $T_{W1}^{-1}\, T\, T_{W2}$ is the corresponding transformation between the world coordinate spaces.

3 RMS Deviation

Consider two transformations, T_1 and T_2, each mapping volume A to volume B (the reference volume). Now a point in volume A, x_A (a three vector), is mapped to some point in volume B by each transformation. If the transformations are identical then the points in volume B will be the same. However, in general the transformations differ and so map x_A to two different points: x_{B1} and x_{B2}. The vector difference between these points, $\Delta x = x_{B2} - x_{B1}$, represents the error in the transformation. It is the average magnitude of this error that is of interest.

As world coordinates are being used, the magnitude of the error vector, $|\Delta x|$ is the magnitude of the deviation expressed in millimetres. Therefore some average of this quantity will express the desired average error. Here the root mean square measure is chosen as the desired average since it is easier to deal with analytically.

In addition, the volume of interest for the average must be defined. For measuring the average error over the brain volume a spherical volume is the simplest approximation. A cubic volume is obviously a worse approximation and will tend to be dominated by the error near the corners, which for a rotational error increases with the distance from the centre of the cube.

Using homogeneous coordinates[1] the error vector can be written in matrix form as:

$$\begin{aligned} \Delta x &= M_A x_A && (1)\\ M_A &= T_2 - T_1 && (2) \end{aligned}$$

or as

$$\begin{aligned} \Delta x &= M_B x_B && (3)\\ M_B &= T_2 T_1^{-1} - I && (4) \end{aligned}$$

The form depends on whether the error is a function of the coordinate in volume A or volume B.

In general, the general form $\Delta x = Mx$ can be used to give the squared error:

$$\begin{aligned} E^2 &= |\Delta x|^2 && (5)\\ &= (\Delta x)^\top (\Delta x) && (6)\\ &= x^\top M^\top M x. && (7) \end{aligned}$$

The normalised RMS error is then given by:

$$E_{RMS}^2 = \frac{\int_V E^2\, dx}{\int_V\, dx}. \tag{8}$$

Expanding equation 7 for spherical coordinates, $x = r\hat{x}$ where $\hat{x} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, gives:

$$E^2 = r^2 \hat{x}^\top A^\top A \hat{x} + 2rt^\top A\hat{x} + t^\top t \tag{9}$$

where the 4 by 4 matrix M is decomposed into a 3 by 3 matrix A, and a 3 by 1 vector t, such that:

$$M = \left[\begin{array}{cc} A & t \\ 0\,0\,0 & 0 \end{array} \right]. \tag{10}$$

Now integrating over the desired spherical volume $r \in [0, R]; \theta \in [0, \pi]; \phi \in [0, 2\pi]$ gives:

$$\begin{aligned} E_{RMS}^2 &= \frac{1}{V} \int_0^R dr \int_0^\pi d\theta \int_0^{2\pi} d\phi\; r^2 \sin\theta\; E^2 && (11)\\ &= \frac{1}{V} \int_0^\pi d\theta \int_0^{2\pi} d\phi\; \sin\theta \left(\int_0^R dr\; r^4 \hat{x}^\top A^\top A \hat{x} + 2r^3 t^\top A \hat{x} + r^2 t^\top t \right) && (12)\\ &= \frac{1}{V} \int_0^\pi d\theta \int_0^{2\pi} d\phi\; \sin\theta \left(\frac{1}{5} R^5 \hat{x}^\top A^\top A \hat{x} + \frac{1}{2} R^4 t^\top A \hat{x} + \frac{1}{3} R^3 t^\top t \right) && (13) \end{aligned}$$

where $V = \frac{4\pi}{3} R^3$ is the spherical volume.

Denoting $\hat{x} = (\hat{x}_1, \hat{x}_2, \hat{x}_3)$ and

$$\overline{p} = \int_0^\pi d\theta \int_0^{2\pi} d\phi\; \sin\theta\; p \tag{14}$$

and combining with

$$\begin{aligned} \int_0^{2\pi} d\phi\; 1 &= 2\pi && (15)\\ \int_0^{2\pi} d\phi\; \sin\phi &= 0 && (16)\\ \int_0^{2\pi} d\phi\; \cos\phi &= 0 && (17)\\ \int_0^{2\pi} d\phi\; \sin\phi\cos\phi &= 0 && (18)\\ \int_0^{2\pi} d\phi\; \sin^2\phi &= \pi && (19) \end{aligned}$$

[1] A homogeneous coordinate here is a four vector comprising the normal three vector followed by a 1; that is, $[xyz1]$.

$$\int_0^{2\pi} d\phi \; \cos^2 \phi \;\; = \;\; \pi \tag{20}$$
$$\int_0^{\pi} d\theta \; \sin\theta \; \cos\theta \;\; = \;\; 0 \tag{21}$$
$$\int_0^{\pi} d\theta \; \sin\theta \cos^2\theta \;\; = \;\; \frac{2}{3} \tag{22}$$
$$\int_0^{\pi} d\theta \; \sin^3\theta \;\; = \;\; \frac{4}{3} \tag{23}$$
$$\tag{24}$$

gives:

$$\overline{x_1 x_1} = \overline{x_2 x_2} = \overline{x_3 x_3} \;\; = \;\; \frac{4\pi}{3} \tag{25}$$
$$\overline{x_1} = \overline{x_2} = \overline{x_3} \;\; = \;\; 0 \tag{26}$$
$$\overline{x_1 x_2} = \overline{x_1 x_3} = \overline{x_2 x_3} = \overline{x_2 x_1} = \overline{x_3 x_1} = \overline{x_3 x_2} \;\; = \;\; 0 \tag{27}$$

Furthermore, by expanding the matrix product,

$$\hat{x}^\top Q \hat{x} = \sum_{ij} Q_{ij} \hat{x}_i \hat{x}_j \tag{28}$$

so that

$$\overline{\hat{x}^\top Q \hat{x}} \;\; = \;\; \sum_{ij} Q_{ij} \overline{\hat{x}_i \hat{x}_j} \tag{29}$$
$$= \;\; \frac{4\pi}{3} \sum_i Q_{ii} \tag{30}$$
$$= \;\; \frac{4\pi}{3} \text{Trace}(Q) \tag{31}$$

where the elements Q_{ij} are constants.

Therefore, by substituting the above results, the RMS error is given by:

$$E^2_{RMS} \;\; = \;\; \left(\frac{4\pi}{3} R^3 \right)^{-1} \left(\frac{4\pi}{3} R^3 t^\top t + \frac{4\pi}{15} R^5 \text{Trace}(A^\top A) \right) \tag{32}$$
$$= \;\; \frac{1}{5} R^2 \text{Trace}(A^\top A) + t^\top t \tag{33}$$

3.1 Defining the Centre of the Volume

In the above formula it is assumed that the volume of integration is centred about $x = 0$. If this is not the case (when the origin of the world coordinates is not at the centre of the brain) then the matrix M can be simply modified to include a general centre.

That is, let x_c be the desired centre, and define a new coordinate

$$\tilde{x} = x - x_c. \tag{34}$$

This new coordinate is now zero when $x = x_c$, so that the above derivation is valid for $\tilde{x}$. Furthermore, this translation can be expressed in matrix form (using homogeneous coordinates) as $\tilde{x} = M_c x$ where

$$M_c = \left[\begin{array}{cc} I & -x_c \\ 0\,0\,0 & 1 \end{array} \right].$$

Therefore, $\Delta x = M\, M_c^{-1} \tilde{x}$, giving $\tilde{M} = M\, M_c^{-1}$. This implies that $\tilde{A} = A$ and $\tilde{t} = t + A\, x_c$ so that

$$E^2_{RMS} = \frac{1}{5} R^2 \text{Trace}(A^\top A) + (t + A\, x_c)^\top (t + A\, x_c). \tag{35}$$

4 Summary

The appropriate equations to use are:

$$E_{RMS} = \sqrt{\frac{1}{5} R^2 \mathrm{Trace}(A^\top A) + (t + A\,x_c)^\top (t + A\,x_c)}. \tag{36}$$

where

$$\begin{aligned} M &= T_2\,T_1^{-1} - I & (37) \\ &= \begin{bmatrix} A & t \\ 0\,0\,0 & 0 \end{bmatrix}. & (38) \end{aligned}$$

x_c is the centre of the volume of interest; T_1 and T_2 are the transformations (from initial to reference volume) that are being compared. Furthermore, all transformations must be world to world coordinate transformations.

Note that this implies the integration in the space of the reference volume. To use the initial volume instead, then use $M = T_2 - T_1$, and use x_c as the world centre in the initial volume.